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Differential Equations for Feynman Graph Amplitudes.

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Abstract

It is by now well established that, by means of the integration by part identities [1], all the integrals occurring in the evaluation of a Feynman graph of given topology can be expressed in terms of a few independent master integrals. It is shown in this paper that the integration by part identities can be further used for obtaining a linear system of first order differential equations for the master integrals themselves. The equations can then be used for the numerical evaluation of the amplitudes as well as for investigating their analytic properties, such as the asymptotic and threshold behaviours and the corresponding expansions (and for analytic integration purposes, when possible).

The new method is illustrated through its somewhat detailed application to the case of the one loop self-mass amplitude, by explicitly working out expansions and quadrature formulas, both in arbitrary continuous dimension n and in the $n \rightarrow 4$ limit.

It is then shortly discussed which features of the new method are expected to work in the more general case of multi-point, multi-loop amplitudes.

PACS 11.10.-z Field theory

PACS 11.10.Kk Field theories in dimensions other than four

PACS 11.15.Bt General properties of perturbation theory

1 Introduction

The usefulness of the integration by parts identities [1] for reducing dramatically the number of independent integrals to be evaluated in the calculation of multi-loop Feynman graph amplitudes, expressing them in terms of a small set of independent integrals (often called master integrals), is by now well established (see for instance Ref. [2]). It is the purpose of this paper to point out that the same technique can be used for obtaining a linear system of first order differential equations in any of the external Mandelstam variables for the master integrals themselves [3].

The new differential equations provide with a fairly complete information on the analytic properties of the integrals. That information can be conveniently used, for instance, for the direct numerical evaluation of the integrals themselves, or for the analytic expansions around particular values of the external scalar variables or of the internal masses – and that by merely algebraic methods, without any attempt at carrying out the loop integrations occurring in the definition of the Feynman graph. The equations can also be exploited for attempting their explicit analytic integration (an achievement which is however expected to be possible only in a limited number of particular cases). As required by the integration by part algorithm, all the loop integrals are defined in n continuous dimensions; the $n \rightarrow 4$ limit is also easily worked out within the new approach.

To give a flavour of the features of the method, the almost elementary cases of 1-loop vacuum (0-point) and self-mass (2-point) amplitudes is discussed in some details (in those cases the system of equations reduces actually to a single equation). While it is claimed that the approach can be extended, without significant loss of effectiveness, to more general multi-loop and multi-point amplitudes, it is obvious that the 1-loop 2-point case is too simple to be really significant, so that from this point of view this paper can be considered just an introduction to further work (which is in progress). In any case, it is hoped that the particular application of the new method presented in this paper might be of interest on its own, at least pedagogically.

The paper is articulated in seven Sections. This is the first Section; Section 2 sets the notation and deals with the 1-loop vacuum amplitude; Section 3 establishes the differential equation for the 1-loop self-mass amplitude, which is the main result of the paper; Section 4 discusses a number of expansions which can be worked out from the equation; Section 5 recasts the equation as a quadrature formula; Section 6 is devoted to the $n \rightarrow 4$ limit; Section 7, finally, contains an outlook to the possible extensions of the method to multi-point, multi-loop amplitudes.

2 The 1-loop vacuum amplitude.

We will use a Euclidean n -dimensional loop variable k , with volume element defined as

$$\frac{d^n k}{(2\pi)^{n-2}} ; \quad (1)$$

the Minkowski loop variables are recovered by the Wick rotation

$$\int d^n k = \int d^{n-1} k dk_n = -i \left(\int d^{n-1} k dk_0 \right)_{\text{Mink}} = -i \left(\int d^n k \right)_{\text{Mink}} .$$

The simplest 1-loop 0-point graph is shown in Fig.1; the associated amplitudes are

$$T(n, m^2, -\alpha) = \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{(k^2 + m^2)^\alpha} , \quad (2)$$

where α is a positive integer.

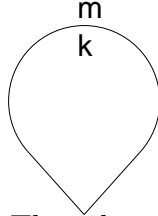


Fig.1: The 1-loop vacuum graph.

An alternative dimensionless definition is

$$T(n, (m/\mu)^2, -\alpha) = \mu^{2\alpha-n} \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{(k^2 + m^2)^\alpha} , \quad (3)$$

where μ is a scale with the dimension of a mass. For simplicity, we will however use in the following Eq.(2), corresponding to Eq.(3) at $\mu = 1$. The integration by parts identities are obtained starting from

$$\int \frac{d^n k}{(2\pi)^{n-2}} \frac{\partial}{\partial k_\mu} \left(k_\mu \frac{1}{(k^2 + m^2)^\alpha} \right) = 0 ,$$

(the integral of the divergence vanishes for small enough n); carrying out the algebra, one obtains

$$T(n, m^2, -\alpha) = \left(1 - \frac{n}{2(\alpha - 1)} \right) \frac{1}{m^2} T(n, m^2, -\alpha + 1) , \quad (4)$$

or the equivalent equation

$$T(n, m^2, -\alpha) = \frac{2\alpha}{2\alpha - n} m^2 T(n, m^2, -\alpha - 1) . \quad (5)$$

With Eq.(4) all the integrals with $\alpha > 1$ are expressed in terms of the “master integral” corresponding to $\alpha = 1$,

$$T(n, m^2) = T(n, m^2, -1) = \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{k^2 + m^2} , \quad (6)$$

while the continuation of Eq.(5) to $\alpha = 0$ gives

$$T(n, m^2, 0) = \int \frac{d^n k}{(2\pi)^{n-2}} = 0$$

(the same applies to negative values of α).

$T(n, m^2)$ is divergent in the $n \rightarrow 4$ limit; by using twice Eq.(4) one gets

$$T(n, m^2) = \frac{8m^4}{(n-2)(n-4)} T(n, m^2, -3) , \quad (7)$$

which provides with a convenient analytic continuation in n , as $T(n, m^2, -3)$ is finite at $n = 4$. According to the definition Eq.(2), scaling k by m and then expressing the loop momentum in n -spherical coordinates, $d^n k = m^n K^{n-1} dK d\Omega(n)$ one easily obtains

$$T(n, m^2, -3) = \frac{m^{n-6} \Omega(n)}{(2\pi)^{n-2}} \int_0^\infty \frac{K^{n-1} dK}{(K^2 + 1)^3} , \quad (8)$$

where $\Omega(n)$ is the n -dimensional solid angle. At $n = 4$, a simple explicit calculation gives

$$T(4, m^2, -3) = \frac{1}{8m^2} ;$$

one can then write

$$T(n, m^2) = \frac{m^{n-2}}{(n-2)(n-4)} C(n) , \quad (9)$$

where $C(n)$ is a suitable dimensionless function of n , defined as

$$C(n) = 8T(n, 1, -3) , \quad (10)$$

with the known limiting value at $n = 4$

$$C(4) = 1 . \quad (11)$$

In most applications $C(n)$ can be kept as an overall factor; its expansion in n around $n = 4$, which can be obtained from Eq.s(10,8), is actually not needed if $C(n)$ multiplies a finite expression in which the $1/(n - 4)$ singularities have cancelled out.

It is further to be noted that for $n > 2$ Eq.(9) implies

$$T(n, 0) = 0 . \quad (12)$$

For convenience of later use, let us also look at the 1-loop, 2-denominator vacuum graph of Fig.2.

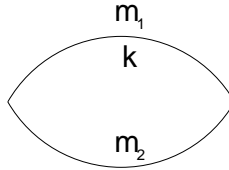


Fig.2: The 1-loop, 2-denominator vacuum graph.

Considering for simplicity only the case in which both denominators are raised to the first power, by partial fractioning one finds

$$\int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{(k^2 + m_1^2)(k^2 + m_2^2)} = \frac{1}{m_2^2 - m_1^2} \left(T(n, m_1^2) - T(n, m_2^2) \right) , \quad (13)$$

showing that the amplitudes associated to the graph of Fig.2 are just algebraic combinations of the amplitudes $T(n, m^2, \alpha)$ associated to Fig.1.

3 The 1-loop self-mass amplitude.

The 1-loop 2-point (self-mass) graph is shown in Fig.3.

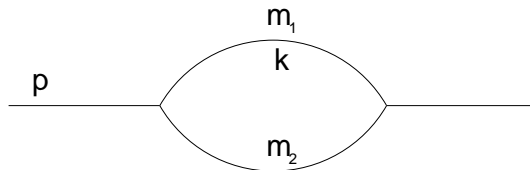


Fig.3: The 1-loop self-mass graph.

The associated scalar integrals are

$$S(n, m_1^2, m_2^2, -\alpha_1, -\alpha_2, p^2) = \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{(k^2 + m_1^2)^{\alpha_1} ((p-k)^2 + m_2^2)^{\alpha_2}} , \quad (14)$$

where α_1, α_2 are positive integers and the external vector p is also Euclidean ($p^2 > 0$ if p is spacelike; the timelike region is to be recovered by means of the usual analytic continuation). The integration by parts identities are derived from the equations

$$\int \frac{d^n k}{(2\pi)^{n-2}} \frac{\partial}{\partial k_\mu} \left(v_\mu \frac{1}{(k^2 + m_1^2)^{\alpha_1} ((p-k)^2 + m_2^2)^{\alpha_2}} \right) = 0 , \quad (15)$$

where the vector v_μ can be either p_μ or k_μ , so that there are 2 identities for each choice of the pair α_1, α_2 . Writing down explicitly the identities is elementary; as a consequence of the identities, it turns out that all the integrals defined in Eq.(14) with α_1 or α_2 larger than 1 can be expressed in terms of a single master integral corresponding to $\alpha_1 = \alpha_2 = 1$, namely

$$S(n, m_1^2, m_2^2, p^2) = \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{(k^2 + m_1^2)((p-k)^2 + m_2^2)} . \quad (16)$$

Indeed, by writing explicitly the two identities corresponding to $\alpha_1 = \alpha_2 = 1$ in Eq.(15), after some elementary algebra one finds

$$\begin{aligned} S(n, m_1^2, m_2^2, -1, -2, p^2) &= \frac{1}{[p^2 + (m_1 + m_2)^2][p^2 - (m_1 + m_2)^2]} \\ &\cdot \left[(n-3)(m_1^2 - m_2^2 - p^2)S(n, m_1^2, m_2^2, p^2) + (n-2)T(n, m_1^2) \right. \\ &\quad \left. - (n-2)\frac{p^2 + m_1^2 + m_2^2}{2m_2^2}T(n, m_2^2) \right] . \end{aligned} \quad (17)$$

The similar formula for $S(n, m_1^2, m_2^2, -2, -1, p^2)$ is then immediately obtained by exchanging m_1 with m_2 in Eq.(17); by further using the relation

$$S(n, m_1^2, m_2^2, -(\alpha_1 + 1), -\alpha_2, p^2) = -\frac{1}{\alpha_1} \frac{\partial}{\partial m_1^2} S(n, m_1^2, m_2^2, -\alpha_1, -\alpha_2, p^2) , \quad (18)$$

which follows at once from the definition Eq.(14), one obtains the corresponding equations for higher values of α_1, α_2 .

Summing up: all the integrals $S(n, m_1^2, m_2^2, -\alpha_1, -\alpha_2, p^2)$, Eq.(14), can be expressed as a combination of the self-mass master integral $S(n, m_1^2, m_2^2, p^2)$ and the vacuum master integral $T(n, m^2)$, defined in Eq.(16) and Eq.(6), times the ratio of suitable polynomials in p^2 , m_1^2 and m_2^2 — a well established and already known result.

We consider now the p^2 -derivative of $S(n, m_1^2, m_2^2, p^2)$. We can obviously write

$$p^2 \frac{\partial}{\partial p^2} S(n, m_1^2, m_2^2, p^2) = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} S(n, m_1^2, m_2^2, p^2) . \quad (19)$$

By using the definition of $S(n, m_1^2, m_2^2, p^2)$, Eq.(16), and then carrying out explicitly the p_μ -derivatives, one obtains

$$p_\mu \frac{\partial}{\partial p_\mu} S(n, m_1^2, m_2^2, p^2) = -2 \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{k^2 + m_1^2} \frac{p^2 - p \cdot k}{((p - k)^2 + m_2^2)^2} ;$$

with the obvious identity

$$p \cdot k = \frac{1}{2} \left[(k^2 + m_1^2) - ((p - k)^2 + m_2^2) + p^2 - m_1^2 + m_2^2 \right] ,$$

the previous equation becomes

$$\begin{aligned} p_\mu \frac{\partial}{\partial p_\mu} S(n, m_1^2, m_2^2, p^2) = & - (p^2 + m_1^2 - m_2^2) S(n, m_1^2, m_2^2, -1, -2, p^2) \\ & - S(n, m_1^2, m_2^2, p^2) + T(n, m_2^2, -2) ; \end{aligned}$$

by using Eq.(4) and Eq.(17), after minor rearrangements Eq.(19) can be written as

$$\begin{aligned} \frac{\partial}{\partial p^2} S(n, m_1^2, m_2^2, p^2) = & - \frac{n-2}{2p^2} S(n, m_1^2, m_2^2, p^2) \\ & + (n-3) \frac{p^2 + m_1^2 + m_2^2}{[p^2 + (m_1 + m_2)^2][p^2 + (m_1 - m_2)^2]} S(n, m_1^2, m_2^2, p^2) \\ & - (n-2) \frac{(p^2 + m_1^2 - m_2^2)T(n, m_1^2) + (p^2 - m_1^2 + m_2^2)T(n, m_2^2)}{2p^2[p^2 + (m_1 + m_2)^2][p^2 + (m_1 - m_2)^2]} , \end{aligned} \quad (20)$$

That same equation can be derived, alternatively, starting from the definition of $S(n, m_1^2, m_2^2, p^2)$ Eq.(16) and replacing the loop variable k by k/λ , obtaining

$$S(n, m_1^2, m_2^2, p^2) = \lambda^{4-n} S(n, \lambda^2 m_1^2, \lambda^2 m_2^2, \lambda^2 p^2) ;$$

by acting on both sides with $\lambda^2 \frac{\partial}{\partial \lambda^2}$ one gets the familiar scaling equation

$$\left(p^2 \frac{\partial}{\partial p^2} + m_1^2 \frac{\partial}{\partial m_1^2} + m_2^2 \frac{\partial}{\partial m_2^2} + \frac{4-n}{2} \right) S(n, m_1^2, m_2^2, p^2) = 0 ,$$

which according to Eq.(18) can also be written as

$$\begin{aligned} p^2 \frac{\partial}{\partial p^2} S(n, m_1^2, m_2^2, p^2) = & \frac{n-4}{2} S(n, m_1^2, m_2^2, p^2) \\ & + S(n, m_1^2, m_2^2, -2, -1, p^2) + S(n, m_1^2, m_2^2, -1, -2, p^2) ; \end{aligned}$$

by substituting Eq.(17) and the similar formula for $S(n, m_1^2, m_2^2, -2, -1, p^2)$, Eq.(20) is recovered.

Eq.(20) is the central result of this paper. It is an inhomogeneous, first-order differential equation in the external variable p^2 for the master integral $S(n, m_1^2, m_2^2, p^2)$; the coefficients are ratios of polynomials in p^2 and the masses, the inhomogeneous term is known, consisting of the simpler vacuum amplitudes $T(n, m^2)$ discussed at length in the previous section. As Eq.(20) is a first order equation in p^2 , it determines $S(n, m_1^2, m_2^2, p^2)$ once a single “initial value” for some value of the variable p^2 is provided; given that initial value, the numerical integration of the equation up to any desired value of p^2 can be easily carried out along any path in the complex p^2 -plane — and that for arbitrary continuous dimension n , the $n \rightarrow 4$ limit being just a particular case (see below).

Eq.(20) can also be used for studying the analytic properties of $S(n, m_1^2, m_2^2, p^2)$, its value at specific values of p^2 or, more generally, its expansion around those values. Eq.(20) is particularly well suited for the discussion of the properties of $S(n, m_1^2, m_2^2, p^2)$ at the potentially singular points of the equation, namely $p^2 = 0$, $p^2 = -(m_1 - m_2)^2$, $p^2 = -(m_1 + m_2)^2$ and $p^2 \rightarrow \infty$; Eq.(20), in addition, can also be recast in the form of a quadrature formula which can then be used for the actual analytical integration in the $n \rightarrow 4$ limit.

4 Expansions of the 1-loop self-mass amplitude.

Let us start to discuss the point $p^2 = 0$; to that aim, Eq.(20) can be rewritten in the fully equivalent way

$$\begin{aligned} \frac{\partial}{\partial p^2} S(n, m_1^2, m_2^2, p^2) = & -\frac{n-2}{2p^2} \left(S(n, m_1^2, m_2^2, p^2) + \frac{T(n, m_1^2) - T(n, m_2^2)}{m_1^2 - m_2^2} \right) \\ & + \frac{n-3}{2} \left(\frac{1}{p^2 + (m_1 + m_2)^2} + \frac{1}{p^2 + (m_1 - m_2)^2} \right) S(n, m_1^2, m_2^2, p^2) \\ & + \frac{n-2}{4m_1(m_1^2 - m_2^2)} \left(\frac{m_1 - m_2}{p^2 + (m_1 + m_2)^2} + \frac{m_1 + m_2}{p^2 + (m_1 - m_2)^2} \right) T(n, m_1^2) \\ & + \frac{n-2}{4m_2(m_1^2 - m_2^2)} \left(\frac{m_1 - m_2}{p^2 + (m_1 + m_2)^2} - \frac{m_1 + m_2}{p^2 + (m_1 - m_2)^2} \right) T(n, m_2^2) . \end{aligned} \quad (21)$$

By inspection of the definition Eq.(16) (but without any explicit integration on the loop momentum k) $S(n, m_1^2, m_2^2, p^2)$ is seen to be regular at $p^2 = 0$, *i.e.* the function and

its derivatives are finite at that point. The apparent singularity of Eq.(21) at $p^2 = 0$ must therefore disappear: that amounts to the condition

$$S(n, m_1^2, m_2^2, 0) = -\frac{T(n, m_1^2) - T(n, m_2^2)}{m_1^2 - m_2^2}, \quad (22)$$

where $T(n, m^2)$ is known, as already remarked. In other words, the equation itself gives the explicit value of $S(n, m_1^2, m_2^2, 0)$, once the information of the regularity at $p^2 = 0$ is provided. That same result, Eq.(22) may be also obtained by noting that at $p_\mu = 0$ (implying $p^2 = 0$) Eq.(16) goes into Eq.(13).

Knowing that $S(n, m_1^2, m_2^2, p^2)$ is regular at $p^2 = 0$, we can also expand $S(n, m_1^2, m_2^2, p^2)$ in p^2 around that point

$$S(n, m_1^2, m_2^2, p^2) = \sum_{k=0}^{\infty} S_k(n, m_1^2, m_2^2)(p^2)^k, \quad (23)$$

where $S_0(n, m_1^2, m_2^2) = S(n, m_1^2, m_2^2, 0)$ is of course given by Eq.(22). By inserting the expansion Eq.(23) into Eq.(21) the coefficients $S_k(n, m_1^2, m_2^2)$ are recursively determined; the first explicit values are

$$\begin{aligned} S_1(n, m_1^2, m_2^2) &= \frac{1}{n(m_1^2 - m_2^2)^3} \left[-(n-4)(m_1^2 + m_2^2) (T(n, m_1^2) - T(n, m_2^2)) \right. \\ &\quad \left. + 2(n-2) (m_2^2 T(n, m_1^2) - m_1^2 T(n, m_2^2)) \right], \\ S_2(n, m_1^2, m_2^2) &= \frac{1}{n(n+2)(m_1^2 - m_2^2)^5} \\ &\quad \left[(n-4) (6(m_1^2 + m_2^2)^2 - n(m_1^2 - m_2^2)^2) (T(n, m_1^2) - T(n, m_2^2)) \right. \\ &\quad \left. - 12(n-2)(m_1^2 + m_2^2) (m_2^2 T(n, m_1^2) - m_1^2 T(n, m_2^2)) \right]. \end{aligned} \quad (24)$$

Note that the knowledge of the coefficients of the expansion around $p^2 = 0$ allows also the accurate evaluation of the *r.h.s* of Eq.(21) for small values of p^2 when trying its numerical integration starting from $p^2 = 0$.

A similar approach can be in principle worked out also at the point $p^2 = -(m_1 - m_2)^2$. It is known, from the conventional study of the analytic properties of $S(n, m_1^2, m_2^2, p^2)$ (which will not be repeated here), that $S(n, m_1^2, m_2^2, p^2)$ is also regular at that particular value of p^2 (sometimes referred to as pseudo-threshold). Imposing the regularity at that point Eq.(21) gives

$$S(n, m_1^2, m_2^2, -(m_1 - m_2)^2) = -\frac{n-2}{n-3} \frac{m_2 T(n, m_1^2) - m_1 T(n, m_2^2)}{2m_1 m_2 (m_1 - m_2)}. \quad (25)$$

It is a somewhat striking feature of this approach that the condition of regularity at $p^2 = 0$, determining completely $S(n, m_1^2, m_2^2, p^2)$ through Eq.(21), which is a first order differential equation, implies also the regularity at $p^2 = -(m_1 - m_2)^2$, a result which we were however unable to derive directly from Eq.(21) only, *i.e.* without explicit reference to the definition Eq.(16).

For discussing the $p^2 \rightarrow \infty$ limit, let us switch to the usual inverse variable $y = 1/p^2$, introducing the function

$$\Phi(n, m_1^2, m_2^2, y) = S\left(n, m_1^2, m_2^2, \frac{1}{y}\right). \quad (26)$$

In terms of the new variable y , Eq.(21) becomes

$$\begin{aligned} \frac{\partial}{\partial y} \Phi(n, m_1^2, m_2^2, y) = & -\frac{n-4}{2y} \Phi(n, m_1^2, m_2^2, y) \\ & + \frac{n-3}{2} \left(\frac{1}{y + \frac{1}{(m_1+m_2)^2}} + \frac{1}{y + \frac{1}{(m_1-m_2)^2}} \right) \Phi(n, m_1^2, m_2^2, y) \\ & + \frac{n-2}{4m_1(m_1^2 - m_2^2)} \left(\frac{m_1 - m_2}{y + \frac{1}{(m_1+m_2)^2}} + \frac{m_1 + m_2}{y + \frac{1}{(m_1-m_2)^2}} \right) T(n, m_1^2) \\ & + \frac{n-2}{4m_2(m_1^2 - m_2^2)} \left(\frac{m_1 - m_2}{y + \frac{1}{(m_1+m_2)^2}} - \frac{m_1 + m_2}{y + \frac{1}{(m_1-m_2)^2}} \right) T(n, m_2^2). \quad (27) \end{aligned}$$

The equation exhibits at $y = 0$ the singular behaviour $y^{-(n-4)/2}$, so that the function must be expanded for small y as the sum of two series corresponding to the singular and regular parts

$$\Phi(n, m_1^2, m_2^2, y) = y^{-(n-4)/2} \sum_{k=0}^{\infty} \Phi_k^{(s)}(n, m_1^2, m_2^2) y^k + \sum_{k=0}^{\infty} \Phi_k^{(r)}(n, m_1^2, m_2^2) y^k. \quad (28)$$

According to Eq.s(26,28), the asymptotic expansion of $S(n, m_1^2, m_2^2, p^2)$ for large p^2 is also given by

$$S(n, m_1^2, m_2^2, p^2) = (p^2)^{(n-4)/2} \sum_{k=0}^{\infty} \Phi_k^{(s)}(n, m_1^2, m_2^2) \frac{1}{(p^2)^k} + \sum_{k=0}^{\infty} \Phi_k^{(r)}(n, m_1^2, m_2^2) \frac{1}{(p^2)^k}. \quad (29)$$

By inserting the expansion Eq.(28) into Eq.(27) – or more directly the expansion Eq.(29) into Eq.(21), disregarding from now on the variable y – and imposing the equality of the

coefficients of the same powers of the expansion parameter one obtains

$$\begin{aligned}\Phi_0^{(r)}(n, m_1^2, m_2^2) &= 0 , \\ \Phi_1^{(r)}(n, m_1^2, m_2^2) &= T(n, m_1^2) + T(n, m_2^2) , \\ \Phi_2^{(r)}(n, m_1^2, m_2^2) &= \left(\frac{n-4}{n}m_1^2 - m_2^2\right) T(n, m_1^2) + \left(\frac{n-4}{n}m_2^2 - m_1^2\right) T(n, m_2^2) ,\end{aligned}\quad (30)$$

and

$$\begin{aligned}\Phi_1^{(s)}(n, m_1^2, m_2^2) &= (n-3)(m_1^2 + m_2^2)\Phi_0^{(s)}(n, m_1^2, m_2^2) , \\ \Phi_2^{(s)}(n, m_1^2, m_2^2) &= \frac{1}{2}(n-3) \left[(n-4)(m_1^4 + m_2^4) + 2(n-6)m_1^2 m_2^2 \right] \Phi_0^{(s)}(n, m_1^2, m_2^2)\end{aligned}\quad (31)$$

Note that Eq.(21), by itself, is unable to fix completely all the coefficients of the expansion; the overall normalization of the singular part, $\Phi_0^{(s)}(n, m_1^2, m_2^2)$ is in fact as yet not determined. But some more information on $\Phi_0^{(s)}(n, m_1^2, m_2^2)$ can be obtained from Eq.(17), by writing it, according to Eq.(18), in the form

$$\begin{aligned}\frac{\partial}{\partial m_2^2} S(n, m_1^2, m_2^2, p^2) &= -\frac{1}{[p^2 + (m_1 + m_2)^2][p^2 - (m_1 + m_2)^2]} \\ &\quad \left[(n-3)(m_1^2 - m_2^2 - p^2)S(n, m_1^2, m_2^2, p^2) + (n-2)T(n, m_1^2) \right. \\ &\quad \left. - (n-2)\frac{p^2 + m_1^2 + m_2^2}{2m_2^2}T(n, m_2^2) \right] ;\end{aligned}\quad (32)$$

by inserting in it the large- p^2 expansion Eq.(29) and then equating the coefficients of the same powers of p^2 we obtain the additional equation

$$\frac{\partial}{\partial m_2^2} \Phi_0^{(s)}(n, m_1^2, m_2^2) = 0 ;\quad (33)$$

as the same is true for the derivative with respect to m_1^2 , we can write as a conclusion

$$\Phi_0^{(s)}(n, m_1^2, m_2^2) = \Phi_0^{(s)}(n) ,\quad (34)$$

expressing the fact that the still unknown leading coefficient $\Phi_0^{(s)}(n)$ is independent of the masses, but depends only on the dimension n .

Due to the condition $\Phi_0^{(r)}(n, m_1^2, m_2^2) = 0$, Eq.(30) and of the expansion Eq.(29), for $n > 2$ Eq.(34) implies

$$\lim_{p^2 \rightarrow \infty} (p^2)^{-(n-4)/2} S(n, m_1^2, m_2^2, p^2) = \Phi_0^{(s)}(n) ,\quad (35)$$

while for smaller values of n the dominant term of the expansion is $\Phi_1^{(r)}(n, m_1^2, m_2^2)/(p^2)$.

The point $p^2 = -(m_1 + m_2)^2$, finally, corresponds to the usual threshold and is therefore another singular point for Eq.(21) and $S(n, m_1^2, m_2^2, p^2)$. It can be discussed, when required, along the lines sketched for $p^2 \rightarrow \infty$. In analogy to Eq.(29) one can write the expansion around $p^2 = -(m_1 + m_2)^2$ as the sum of two series,

$$\begin{aligned} S(n, m_1^2, m_2^2, p^2) &= (p^2 + (m_1 + m_2)^2)^{(n-3)/2} \sum_{k=0}^{\infty} \Psi_k^{(s)}(n, m_1^2, m_2^2) (p^2 + (m_1 + m_2)^2)^k \\ &+ \sum_{k=0}^{\infty} \Psi_k^{(r)}(n, m_1^2, m_2^2) (p^2 + (m_1 + m_2)^2)^k ; \end{aligned} \quad (36)$$

by inserting the expansion in Eq.(21), it turns out that the coefficients $\Psi_k^{(r)}(n, m_1^2, m_2^2)$ are completely fixed, while the $\Psi_k^{(s)}(n, m_1^2, m_2^2)$ are all proportional to a same constant, which can be taken to be $\Psi_0^{(s)}(n, m_1^2, m_2^2)$ and is not determined by Eq.(21) alone. The first values of the coefficients are

$$\begin{aligned} \Psi_0^{(r)}(n, m_1^2, m_2^2) &= -\frac{(n-3)(n-2)}{2m_1m_2(m_1+m_2)} (m_2T(n, m_1^2) + m_1T(n, m_2^2)) \\ \Psi_1^{(s)}(n, m_1^2, m_2^2) &= \left(\frac{n-2}{2(m_1+m_2)^2} - \frac{n-3}{8m_1m_2} \right) \Psi_0^{(s)}(n, m_1^2, m_2^2) . \end{aligned} \quad (37)$$

As already done for the large p^2 expansion, one can insert the expansion Eq.(36) and the explicit values Eq.(37) in Eq.(32), obtaining the equation

$$\frac{\partial}{\partial m_2} \Psi_0^{(s)}(n, m_1^2, m_2^2) = \left(\frac{n-3}{2m_2} - \frac{n-2}{m_1+m_2} \right) \Psi_0^{(s)}(n, m_1^2, m_2^2) . \quad (38)$$

Its solution, symmetrized in m_1, m_2 , is

$$\Psi_0^{(s)}(n, m_1^2, m_2^2) = \frac{(m_1m_2)^{(n-3)/2}}{(m_1+m_2)^{n-2}} \Psi_0^{(s)}(n) , \quad (39)$$

where the still unknown coefficient $\Psi_0^{(s)}(n)$ depends only on n .

5 The quadrature formula.

The 1-loop self-mass case is simple enough to allow to recast the differential equation in the form of a quadrature formula. By putting

$$\begin{aligned} S(n, m_1^2, m_2^2, p^2) &= (p^2)^{-(n-2)/2} \left[(p^2 + (m_1 + m_2)^2) (p^2 + (m_1 - m_2)^2) \right]^{(n-3)/2} \\ &\cdot \sigma(n, m_1^2, m_2^2, p^2) , \end{aligned} \quad (40)$$

Eq.(20) becomes

$$\begin{aligned} \frac{\partial}{\partial p^2} \sigma(n, m_1^2, m_2^2, p^2) = & -\frac{1}{2}(p^2)^{(n-4)/2} \left[(p^2 + (m_1 + m_2)^2) (p^2 + (m_1 - m_2)^2) \right]^{-(n-1)/2} \\ & (n-2) \left[(p^2 + m_1^2 - m_2^2) T(n, m_1^2) + (p^2 - m_1^2 + m_2^2) T(n, m_2^2) \right] . \end{aligned} \quad (41)$$

The above equation is equivalent to the quadrature formula

$$\begin{aligned} \sigma(n, m_1^2, m_2^2, p^2) = & \sigma(n, m_1^2, m_2^2, \bar{p}^2) \\ & - \frac{n-2}{2} \int_{\bar{p}^2}^{p^2} dx x^{(n-4)/2} \left[(x + (m_1 + m_2)^2) (x + (m_1 - m_2)^2) \right]^{-(n-1)/2} \\ & \cdot \left[(x + m_1^2 - m_2^2) T(n, m_1^2) + (x - m_1^2 + m_2^2) T(n, m_2^2) \right] , \end{aligned} \quad (42)$$

where \bar{p}^2 is some suitable value of p^2 . As $S(n, m_1^2, m_2^2, p^2)$ is regular at $p^2 = 0$, for $n > 2$ Eq.(40) implies $\sigma(n, m_1^2, m_2^2, 0) = 0$, so that $\bar{p}^2 = 0$ is a natural choice.

For $m_2 = 0$, by using Eq.(12), one has

$$\sigma(n, m^2, 0, p^2) = -\frac{n-2}{2} T(n, m^2) \int_0^{p^2} dx x^{(n-4)/2} (x + m^2)^{-(n-2)} ; \quad (43)$$

which becomes, by using Eq.(9) and scaling x by m^2 within the integrand,

$$\sigma(n, m^2, 0, p^2) = -\frac{C(n)}{2(n-4)} \int_0^{p^2/m^2} dx x^{(n-4)/2} (x + 1)^{-(n-2)} . \quad (44)$$

By comparing Eq.(35), Eq.(40) and Eq.(44) a closed expression for $\Phi_0^{(s)}(n)$ is obtained

$$\Phi_0^{(s)}(n) = -\frac{C(n)}{2(n-4)} \int_0^\infty dx x^{(n-4)/2} (x + 1)^{-(n-2)} . \quad (45)$$

6 The $n \rightarrow 4$ limit.

It is known that in the $n \rightarrow 4$ limit 1-loop amplitudes are singular at most as $1/(n-4)$; from Eq.(9) one has indeed, for $n \rightarrow 4$,

$$T(n, m^2) = m^2 \left[\frac{C(n)}{2(n-4)} + \frac{1}{2} \ln m - \frac{1}{4} + \mathcal{O}(n-4) \right] , \quad (46)$$

where $C(n)$ is given by Eq.s(10,11). If $\ln m$ is perceived as disturbing, replace Eq.(2) with Eq.(3) and $\ln m$ goes into $\ln(m/\mu)$.

Accordingly, $S(n, m_1^2, m_2^2, p^2)$ can be expanded in the $n \rightarrow 4$ limit as

$$S(n, m_1^2, m_2^2, p^2) = \frac{C(n)}{n-4} S_0(m_1^2, m_2^2, p^2) + S_1(m_1^2, m_2^2, p^2) . \quad (47)$$

Eq.(20) is an identity in n ; therefore, it implies a set of independent equations for the various terms of the expansion in $(n-4)$. The part singular in $(n-4)$, *i.e.* the coefficient of $C(n)/(n-4)$, gives

$$\begin{aligned} \frac{\partial}{\partial p^2} S_0(m_1^2, m_2^2, p^2) = - & \left(\frac{1}{p^2} - \frac{1}{2(p^2 + (m_1 + m_2)^2)} - \frac{1}{2(p^2 + (m_1 - m_2)^2)} \right) \\ & \cdot \left(S_0(m_1^2, m_2^2, p^2) + \frac{1}{2} \right) . \end{aligned} \quad (48)$$

On the other hand, we can expand $S(n, m_1^2, m_2^2, 0)$, Eq.(22); that gives, in the notation of Eq.(47),

$$S_0(m_1^2, m_2^2, 0) = -\frac{1}{2} \quad (49)$$

and

$$S_1(m_1^2, m_2^2, 0) = +\frac{1}{4} - \frac{m_1^2 \ln m_1 - m_2^2 \ln m_2}{2(m_1^2 - m_2^2)} . \quad (50)$$

(Again, $\ln m_1$ becomes $\ln(m_1/\mu)$ etc. if the *r.h.s.* of Eq.(14) is multiplied by μ^{4-n}).

With the input Eq.(49), Eq.(48) gives, for any p^2 ,

$$S_0(m_1^2, m_2^2, p^2) = -\frac{1}{2} , \quad (51)$$

independent of p^2 and of the masses.

For discussing the finite part of Eq.(20) in the $n \rightarrow 4$ limit, it can be convenient to write $S_1(m_1^2, m_2^2, p^2)$ as

$$S_1(m_1^2, m_2^2, p^2) = S_1(m_1^2, m_2^2, 0) + \overline{S}(m_1^2, m_2^2, p^2) , \quad (52)$$

where $S_1(m_1^2, m_2^2, 0)$ is known from Eq.(50). Note that Eq.(52), due to Eq.(51), amounts to

$$\overline{S}(m_1^2, m_2^2, p^2) = \lim_{n \rightarrow 4} \left(S(n, m_1^2, m_2^2, p^2) - S(n, m_1^2, m_2^2, 0) \right) , \quad (53)$$

or, in equivalent way,

$$S(n, m_1^2, m_2^2, p^2) \xrightarrow{n \rightarrow 4} S(n, m_1^2, m_2^2, 0) + \overline{S}(m_1^2, m_2^2, p^2) . \quad (54)$$

From Eq.(20) – or, rather, from its equivalent form Eq.(21) – one then obtains for $\overline{S}(m_1^2, m_2^2, p^2)$ the equation

$$\begin{aligned} \frac{\partial}{\partial p^2} \overline{S}(m_1^2, m_2^2, p^2) = & -\frac{1}{p^2} \overline{S}(m_1^2, m_2^2, p^2) \\ & + \frac{1}{2} \left(\frac{1}{p^2 + (m_1 + m_2)^2} + \frac{1}{p^2 + (m_1 - m_2)^2} \right) \overline{S}(m_1^2, m_2^2, p^2) \\ & + \frac{m_1^2 m_2^2}{m_1^2 - m_2^2} \ln \left(\frac{m_1}{m_2} \right) \frac{1}{[p^2 + (m_1 + m_2)^2][p^2 + (m_1 - m_2)^2]} \\ & - \frac{1}{8} \left(\frac{1}{p^2 + (m_1 + m_2)^2} + \frac{1}{p^2 + (m_1 - m_2)^2} \right) . \end{aligned} \quad (55)$$

Eq.(55) can be used, in the same way as Eq.(20), for obtaining the coefficients of the expansions of $\overline{S}(m_1^2, m_2^2, p^2)$ around particular values of p^2 or a quadrature formula.

The expansion at $p^2 = 0$ is straightforward. By writing

$$\overline{S}(m_1^2, m_2^2, p^2) = \overline{S}_1(m_1^2, m_2^2) p^2 + \mathcal{O}((p^2)^2) , \quad (56)$$

Eq.(55) gives

$$\overline{S}_1(m_1^2, m_2^2) = \frac{m_1^2 m_2^2}{2(m_1^2 - m_2^2)^3} \ln \left(\frac{m_1}{m_2} \right) - \frac{m_1^2 + m_2^2}{8(m_1^2 - m_2^2)^2} , \quad (57)$$

which is just the $n \rightarrow 4$ limit of $S_1(n, m_1^2, m_2^2)$, Eq.(24).

More interesting is the $n \rightarrow 4$ limit of the $p^2 \rightarrow \infty$ expansion. Let us start with Eq.(29); for $n \rightarrow 4$ one has

$$(p^2)^{(n-4)/2} \xrightarrow{n \rightarrow 4} 1 + \frac{n-4}{2} \ln p^2 \quad (58)$$

(see the comments to $\ln m_1$ in Eq.(50) for the appearance of $\ln p^2$). As it is known that one loop amplitudes have at most simple poles in $(n-4)$, the leading coefficient $\Phi_0^{(s)}(n)$, Eq.(34), can be expanded as

$$\Phi_0^{(s)}(n) \xrightarrow{n \rightarrow 4} \frac{C(n)}{n-4} \phi_0 + \phi_1 , \quad (59)$$

where $C(n)$ is again given by Eq.s(10,11) and ϕ_0, ϕ_1 are as yet unknown constants. In the $n \rightarrow 4$ limit the large p^2 expansion of Eq.(29) then reads

$$\begin{aligned}
S(n, m_1^2, m_2^2, p^2) &\xrightarrow{n \rightarrow 4} \frac{C(n)}{n-4} \phi_0 + \phi_1 + \frac{1}{2} \phi_0 \ln p^2 \\
&+ \left[(m_1^2 + m_2^2) \left(\frac{C(n)}{n-4} \frac{2\phi_0 + 1}{2} + \phi_0 + \phi_1 - \frac{1}{4} + \frac{1}{2} \phi_0 \ln p^2 \right) \right. \\
&\left. + \frac{1}{2} (m_1^2 \ln m_1 + m_2^2 \ln m_2) \right] \frac{1}{p^2} + \mathcal{O} \left(\frac{1}{(p^2)^2} \right) .
\end{aligned} \tag{60}$$

It is however known from Eq.s(47,51) that the pole in $(n-4)$ is independent of p^2 and of the masses; that implies

$$\phi_0 = -\frac{1}{2} , \tag{61}$$

so that Eq.(60) becomes

$$\begin{aligned}
S(n, m_1^2, m_2^2, p^2) &\xrightarrow{n \rightarrow 4} -\frac{C(n)}{2(n-4)} + \phi_1 - \frac{1}{4} \ln p^2 \\
&+ \left[\left(\phi_1 - \frac{3}{4} \right) (m_1^2 + m_2^2) - \frac{1}{4} m_1^2 \ln \left(\frac{p^2}{m_1^2} \right) - \frac{1}{4} m_2^2 \ln \left(\frac{p^2}{m_2^2} \right) \right] \frac{1}{p^2} + \mathcal{O} \left(\frac{1}{(p^2)^2} \right) ,
\end{aligned} \tag{62}$$

where the constant ϕ_1 is still unknown.

Let us now look at the quadrature formula. We have already seen that the replacement Eq.(40) transforms Eq.(20) into the quadrature formula Eq.(42). One can similarly put

$$\overline{S}(m_1^2, m_2^2, p^2) = \frac{1}{p^2} \sqrt{(p^2 + (m_1 + m_2)^2) (p^2 + (m_1 - m_2)^2)} \cdot \overline{\sigma}(m_1^2, m_2^2, p^2) ; \tag{63}$$

according to that definition, $\overline{\sigma}(m_1^2, m_2^2, 0) = 0$. Eq.(55) then becomes the quadrature formula

$$\begin{aligned}
\overline{\sigma}(m_1^2, m_2^2, p^2) &= \int_0^{p^2} \frac{dx}{[(x + (m_1 + m_2)^2) (x + (m_1 - m_2)^2)]^{\frac{3}{2}}} \\
&\cdot \left[-\frac{1}{4} x (x + m_1^2 + m_2^2) + \frac{m_1^2 m_2^2}{m_1^2 - m_2^2} \ln \left(\frac{m_1}{m_2} \right) \right] .
\end{aligned} \tag{64}$$

An elementary integration gives the known [4] explicit analytic expression

$$\begin{aligned}
\overline{S}(m_1^2, m_2^2, p^2) &= \frac{1}{4} + \frac{m_1^2 + m_2^2}{4(m_1^2 - m_2^2)} \ln \left(\frac{m_1}{m_2} \right) \\
&+ \frac{1}{4p^2} \left[\sqrt{(p^2 + (m_1 + m_2)^2) (p^2 + (m_1 - m_2)^2)} \ln u(p^2) + (m_1^2 - m_2^2) \ln \left(\frac{m_1}{m_2} \right) \right] ,
\end{aligned} \tag{65}$$

where

$$u(p^2) = \frac{\sqrt{p^2 + (m_1 + m_2)^2} - \sqrt{p^2 + (m_1 - m_2)^2}}{\sqrt{p^2 + (m_1 + m_2)^2} + \sqrt{p^2 + (m_1 - m_2)^2}} . \tag{66}$$

For real and positive p^2 , $u(p^2)$ is also real and positive, ranging from $u(+\infty) = 0$ to $u(0) = m_2/m_1$ (if $m_2 < m_1$). For real and negative $p^2 > -(m_1 - m_2)^2$, $u(p^2)$ remains real and positive, with $u(-(m_1 - m_2)^2) = 1$; at $p^2 = -(m_1 - m_2)^2$, further, Eq.(65) reads

$$\overline{S}(m_1^2, m_2^2, -(m_1 - m_2)^2) = \frac{1}{4} + \frac{m_1 m_2}{2(m_1^2 - m_2^2)} \ln \left(\frac{m_1}{m_2} \right) . \quad (67)$$

From Eq.(25) and Eq.(22) a simple explicit calculation gives

$$\lim_{n \rightarrow 4} \left(S(n, m_1^2, m_2^2, -(m_1 - m_2)^2) - S(n, m_1^2, m_2^2, 0) \right) = \frac{1}{4} + \frac{m_1 m_2}{2(m_1^2 - m_2^2)} \ln \left(\frac{m_1}{m_2} \right) , \quad (68)$$

so that Eq.(53) is satisfied at $p^2 = -(m_1 - m_2)^2$.

When p^2 varies from the pseudo-threshold $-(m_1 - m_2)^2$ to the threshold $-(m_1 + m_2)^2$, one can put

$$\sqrt{p^2 + (m_1 - m_2)^2} = i \sqrt{-(p^2 + (m_1 - m_2)^2)} ; \quad (69)$$

$u(p^2)$ then varies on the complex unit circle, $u(p^2) = e^{i\phi(p^2)}$, with $\phi(-(m_1 - m_2)^2) = 0$ and $\phi(-(m_1 + m_2)^2) = -\pi$, so that in Eq.(65) $\ln u(p^2)$ becomes i , the imaginary unit, times an arctangent, the square root in front of it is also imaginary and $\overline{S}(m_1^2, m_2^2, p^2)$ remains real (as $\overline{S}(m_1^2, m_2^2, p^2)$ is analytic at the pseudo-threshold, changing i into $-i$ in Eq.(69) does not change its value). For $p^2 < -(m_1 + m_2)^2$, finally, $u(p^2)$ is negative, ranging from -1 at $p^2 = -(m_1 + m_2)^2$ to 0 at $p^2 = -\infty$; correspondingly, the square root of Eq.(65) is again real, while $\ln u(p^2)$ becomes complex, acquiring an imaginary part $i\pi$ whose sign however can be determined only by giving an infinitesimal imaginary part to p^2 (if $\text{Im} p^2 < 0$, the imaginary part is $i\pi$).

Further, on account of the explicit analytic expression Eq.(65), in the $p^2 \rightarrow \infty$ limit Eq.(54) becomes

$$\begin{aligned} S(n, m_1^2, m_2^2, p^2) &\xrightarrow{n \rightarrow 4} -\frac{C(n)}{2(n-4)} + \frac{1}{2} - \frac{1}{4} \ln p^2 \\ &- \frac{1}{4} \left[m_1^2 + m_2^2 + m_1^2 \ln \left(\frac{p^2}{m_1^2} \right) + m_2^2 \ln \left(\frac{p^2}{m_2^2} \right) \right] \frac{1}{p^2} + \mathcal{O} \left(\frac{1}{(p^2)^2} \right) ; \end{aligned} \quad (70)$$

the comparison with Eq.(62) gives

$$\phi_1 = \frac{1}{2} . \quad (71)$$

Eq.s(61,71) can be obtained also by the explicit integration of Eq.(45) in the $n \rightarrow 4$ limit.

Similarly, one can expand Eq.(65) around the physical threshold $p^2 = -(m_1 + m_2)^2$, obtaining, in the notation of Eq.(39),

$$\Psi_0^{(s)}(4) = -\frac{\pi}{4} . \quad (72)$$

7 Outlook.

As already said in the Introduction, the 1-loop self-mass is too simple for really showing the potential of the new method. It is nevertheless possible to try to assess which features of the method can be reasonably expected to apply also to less elementary cases.

To start with, it is apparent from the derivation that a linear system of first order differential equations can be established, in full generality, for multi-point and multi-loop graphs as well, as the integration by part identities can be written for any number of external lines and loops.

In the case of more general multi-point amplitudes, depending on a set of independent external vectors $p_{i,\mu}$, one has simply to generalize Eq.(19) by considering all the various $p_{i,\mu} \frac{\partial}{\partial p_{j,\mu}}$ combinations and then work out some kinematical algebra for extracting the derivative with respect to any given external scalar variables. A first application to the 1-loop vertex is underway [5].

In the general multi-loop case several independent master integrals are expected to appear, so that rather than a single first order equation there will be a linear system of first order differential equations, whose coefficients however will be in full generality ratios of suitable polynomials in the dimension n , the masses and the external scalar variables (work on the 2-loop “sunrise” self-mass graph is in progress [6]). In the general case, the equations will involve, besides the master integrals for the considered original Feynman graph, a number of “inhomogeneous terms” corresponding to the master integrals for all the related, simpler Feynman graphs (with the same number of loops) obtained by removing a denominator from the original graph; those simpler graph amplitudes are to be considered known — or else can be studied recursively by the same algorithm.

In the general multi-point, multi-loop case and for arbitrary values of the masses simple quadrature formulae and closed analytic results in terms of known functions cannot be expected to exist, but the equations will be anyhow a powerful tool for studying the value of the amplitudes for particular values of the occurring masses as well as for attempting, in particularly simple cases, the analytic integrations.

The equations are particularly well suited for the expansion in $(n-4)$; as the equations are identities in n , the $n \rightarrow 4$ limit can be worked out at once for the whole multi-loop amplitudes, without any reference to the $n \rightarrow 4$ limit of the inserted loops, so that any overlapping divergencies problem is avoided.

Once a system of equations for quantities finite in the $n \rightarrow 4$ limit has been obtained, its numerical integration is straightforward, for virtually any number of loops and external

lines, provided suitable starting points are given. It is reasonable to think (but also confirmed by some preliminary work, [6]) that the absence of the would-be kinematical singularities, such as those at the $p^2 = 0$ for the self-mass, will provide with useful information for the evaluation of those starting points.

The equations will also be very useful, in general, to provide any kind of required expansion of the graph amplitudes, for instance for asymptotically large values of the scalar variables or at the physical thresholds, in terms of very few constants; even if those constants are not expected to be fixed by the equations themselves, the information which they contain can be exploited to make easier the calculation of those missing values.

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